Protective properties and the constrained equal awards rule for claims problems

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Proposed Running Head:
Protective properties and claims problems

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ABSTRACT

We investigate the implications of two protective properties, sustainability and exemption, when imposed separately in conjunction with other basic properties for the resolution of conflicting claims. Under the protective properties, agents with sufficiently small claims in relation to the others are fully reimbursed. We show that the constrained equal awards rule is the only rule satisfying (1) sustainability and claims monotonicity, (2) sustainability and super-modularity, and (3) exemption, order preservation, and bilateral consistency. Then, we extend the notions of the protective properties to groups of agents, and show that no rule satisfies any of these extensions. *Journal of Economic Literature* Classification Numbers: D63, D74.

Keywords: claims problems; sustainability; exemption; constrained equal awards rule.

paper is based on my Ph.D. thesis at the University of Rochester. I am responsible for any remaining deficiency.

1 Introduction

We consider the problem of distributing an infinitely divisible and homogeneous resource among agents having claims on it. An example is when the liquidation value of a bankrupt firm has to be divided among its creditors. A rule is a function that associates with each problem of this kind, called a "claims problem", a division of the amount available, called an "awards vector". How should the division be performed? The literature devoted to the search for the desirable rules is initiated by O'Neill [7].[1]

Our goal is to conduct a systematic analysis of two "protective" properties introduced by Herrero and Villar [5,6]. The properties are intended to fully reimburse agents with relatively small claims. Such preferential treatment is a common phenomenon in the world. For example, think of the interpretation of a claims problem as a particular tax-assessment problem. Here agents' claims correspond to their pre-taxed incomes, and the amount available to "the total amount of their post-taxed incomes" (the difference between the sum of their pre-taxed incomes and the total amount of income taxes). How should agents' post-taxed incomes be assigned?[2] We often observe that agents with relatively low incomes are exempted from income taxation. In other words, their pre-taxed incomes are equal to their post-taxed incomes. The question is how small a claim should be relative to both other claims and the amount available to make the protective treatment desirable.

Herrero and Villar [5,6] formulate two standards of smallness: agent i's claim c_i is "sustainable" if truncating all claims at c_i results in a situation where there is enough to fully reimburse everyone; agent i's claim c_i is "exemptive" if it is not greater than equal division. The first prop-

[1]For a comprehensive survey of this literature, see Thomson [10].

[2]This problem can be understood as the "dual" of the tax-assessment problem considered by Young [15,16]. In that model, agents' claims correspond to their pre-taxed incomes, and the amount available to the total amount of income taxes. Thus, Young's tax-assessment problem focuses on how to allocate the total amount of income taxes. However, our tax-assessment problem emphasizes on how to assign agents' post-taxed incomes.

erty, *sustainability*, says that if an agent's claim is sustainable, he should be fully compensated. The second property, *exemption*, says that if an agent's claim is exemptive, he should be fully compensated. Note that *sustainability* implies *exemption*. However, the converse is not true. For a rule satisfying a variable-population property to be defined shortly, *exemption* implies *sustainability* (Proposition 1).

Herrero and Villar [5,6] find that the constrained equal awards rule[3], which assigns equal amounts to all agents subject to no one receiving more than his claim, satisfies *sustainability*. They then base characterizations of the rule on this property when imposed in conjunction with two other appealing properties. The first one is a composition property. When the amount available decreases from some initial value, there are two ways to look at the situation. We can cancel the initial division and recalculate the awards for the revised amount available. Alternatively, we can take the initial awards calculated on the basis of the initial amount available as claims in dividing the revised amount available. *Composition down* (Moulin [8]) says that both procedures should produce the same awards vector. The second one is the variable-population property alluded to in the previous paragraph. Suppose that an awards vector is chosen for a problem. *Consistency* (Aumann and Maschler [1]; Young [15,16]) says that this awards vector should be in agreement with the awards vector chosen for any problem obtained by imagining some agents leaving with their awards, and re-evaluating the situation from the viewpoint of the remaining agents.[4]

Herrero and Villar [5,6] show that the constrained equal awards rule is the only rule satisfying *sustainability* and *composition down*, and that it is the only rule satisfying *exemption*, *composition down*, and "bilateral consistency"[5]. *Composition down* is an attractive property but unfortu-

[3] For earlier references, see Aumann and Maschler [1] and Dagan [3].

[4] For a comprehensive survey of the literature on *consistency* and its converse, see Thomson [11].

[5] It is a weaker version of *consistency*, which restricts attention to two-agent subgroups.

nately it is not satisfied by many well-known rules, such as the "Talmud rule"[6]. One may wonder whether the characterizations could be obtained by replacing *composition down* with some other property that would be satisfied more generally. We will identify several such properties.

We consider first *claims monotonicity*. It says that when an agent's claim increases, he should not receive less than what he did initially. We show that, surprisingly, this very mild property and *sustainability* together are satisfied only by the constrained equal awards rule (Theorem 1). Next, we turn to two order properties. *Order preservation* (Aumann and Maschler [1]) says that of two agents, the one with the larger claim should not receive less than the other. Also, his loss (the difference between his claim and his award) should not be less than the other's. *Super-modularity* (Dagan, Serrano, and Volij [4]) says that when the amount available increases, of two agents, the one with the larger claim should not receive a smaller share of the increment than the other. We show that *sustainability* and *super-modularity* are satisfied only by the constrained equal awards rule (Theorem 2), and that *exemption*, *order preservation*, and *bilateral consistency* are satisfied only by the rule (Theorem 3).

We replace *composition down* in Theorem 2.1 of Herrero and Villar [5] with *claims monotonicity* or *super-modularity*. In addition, *composition down* in Theorem 2 of Herrero and Villar [6] is replaced with *order preservation*. These replacements are significant since a number of well-known rules satisfy *claims monotonicity*, *super-modularity*, and *order preservation*, but not *composition down*. Examples are the Talmud rule, "Piniles rule", and the "random arrival rule".

In the case of claims problems, suppose that a rule is characterized by a list of properties for problems involving only two agents. In addition, it satisfies *bilateral consistency* and a converse of this property, "converse consistency". Then, its characterization can be extended to more than two agents by imposing *bilateral consistency* or *converse con-*

[6]This rule is defined by Aumann and Maschler [1] to rationalize the recommendations made in the Talmud for several numerical examples.

sistency. Such extensions are immediate consequences of the "Elevator Lemma", which states that if a rule satisfies *bilateral consistency*, and in the two-agent case, coincides with some other rule that satisfies *converse consistency*, then the two rules coincide in general.[7] For instance, the constrained equal awards rule is the only rule satisfying *sustainability* and *composition down* (Herrero and Villar [5] Theorem 2.1). Note that *sustainability* and *exemption* are equivalent in the two-agent case. Thus, the rule is the only rule satisfying *exemption* and *composition down* in that case. Since it satisfies *bilateral consistency* and *converse consistency*, the Elevator Lemma gives us two results: the constrained equal awards rule is the only rule satisfying *exemption*, *composition down*, and *bilateral consistency* (Herrero and Villar [6] Theorem 2), and it is the only rule satisfying *exemption*, *composition down*, and *converse consistency* (Yeh [14] Theorem 2).

Therefore, the Elevator lemma and our new characterizations of the constrained equal awards rule together give us another group of characterizations of the rule: it is the only rule satisfying *exemption*, *claims monotonicity*, and *bilateral consistency* (Proposition 2) or *converse consistency* (Proposition 3), and it is the only rule satisfying *exemption*, *super-modularity*, and *bilateral consistency* (Proposition 4) or *converse consistency* (Proposition 5).

Finally, we formulate versions of *sustainability* and *exemption* for groups by taking the arithmetic average of claims of a group of agents. We show that no rule satisfies any of them (Theorems 4 and 5). Thus, taking the arithmetic average of claims of a group of agents to extend the notions of the protective properties is too demanding.

The paper is organized as follows. In Section 2, we introduce the model, the constrained equal awards rule, and the protective properties.

[7]The lemma is introduced by Thomson [11] and stated in a "model-free" fashion. It says that if a (possibly multi-valued) rule satisfies *consistency* and is a subrule of a *conversely consistent* rule in the two-agent case, then the inclusion holds in general. The lemma as stated here is the special case for single-valued rules. For this expression and a study of the lemma, see Thomson [11].

In Section 3 we present our results and check the independence of the properties appearing in each of our characterizations. In Sections 4 and 5, we state the dual parts of our characterizations, and extend the ideas of the protective properties to groups of agents. Following Section 5 is the conclusion.

2 The model, the constrained equal awards rule, and the protective properties

There is an infinite set of "potential" agents, indexed by the set of natural numbers $\mathbb{N}$. Let $\mathcal{N}$ be the class of finite subsets of $\mathbb{N}$. Given $N \in \mathcal{N}$, an **amount available** $E \in \mathbb{R}_+$ of an infinitely divisible and homogeneous resource has to be distributed among a group of agents N having claims on it.[8] For each $i \in N$, let c_i be **agent $\boldsymbol{i}$'s claim**. Let $c \equiv (c_i)_{i \in N}$ be the claims profile. A **claims problem for $\boldsymbol{N}$** is a pair $(c, E) \in \mathbb{R}^N_+ \times \mathbb{R}_+$ such that $\sum_{i \in N} c_i \geq E$.[9] Let $\mathcal{C}^N$ be the class of claims problems. An **awards vector** for $(c, E) \in \mathcal{C}^N$ is a point $x \in \mathbb{R}^N$ such that $0 \leqq x \leqq c$ and $\sum_{i \in N} x_i = E$.[10] Let $X(c, E)$ be the set of all awards vectors for (c, E). A **rule** is a function defined on $\bigcup_{N \in \mathcal{N}} \mathcal{C}^N$ that associates with each $N \in \mathcal{N}$ and each claims problem $(c, E) \in \mathcal{C}^N$ an awards vector in $X(c, E)$. Our generic notation for rules is φ. For each group $N' \subset N$, we denote $c_{N'} \equiv (c_i)_{i \in N'}$, $\varphi_{N'}(c, E) \equiv (\varphi_i(c, E))_{i \in N'}$, and so on.

We now formally define the constrained equal awards rule and the protective properties.

Constrained equal awards rule, $\boldsymbol{CEA}$: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $i \in N$, $CEA_i(c, E) \equiv \min\{c_i, \lambda\}$, where λ is chosen such that $\sum_{i \in N} CEA_i(c, E) = E$.

[8] By $\mathbb{R}_+$ we denote the set of real numbers, $\mathbb{R}_+ \equiv \{x \in \mathbb{R} \mid x \geq 0\}$.

[9] By $\mathbb{R}^N_+$ we denote the Cartesian product of $|N|$ copies of $\mathbb{R}_+$, indexed by the elements of N.

[10] Vector inequalities: $x \geqq y$, $x \geq y$, and $x > y$.

Sustainability: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $i \in N$, if $\sum_{j \in N} \min\{c_j, c_i\} \leq E$, then $\varphi_i(c, E) = c_i$.

Exemption: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$ and each $i \in N$, if $c_i \leq \frac{E}{|N|}$, then $\varphi_i(c, E) = c_i$.

Note that *sustainability* and *exemption* are equivalent in the two-agent case, and that *sustainability* implies *exemption* in general. Moreover, as we show in the next section, *exemption*, when imposed together with a variable-population property to be defined shortly, implies *sustainability*.

3 The results

The following lemma plays an important role in our presentation. To introduce it, we define two variable-population properties. Consider a problem and suppose that a rule has been chosen to solve the problem. Then, an awards vector is obtained for the problem. Now, imagine that some agents leave with their awards. The first variable-population property says that when the situation is re-evaluated from the viewpoint of the remaining agents, the rule should recommend the same awards for the remaining agents as initially.

Consistency: For each $N \in \mathcal{N}$, each $N' \subset N$, and each $(c, E) \in \mathcal{C}^N$, if $x \equiv \varphi(c, E)$, then $x_{N'} = \varphi\left(c_{N'}, \sum_{i \in N'} x_i\right)$.

A weaker version of *consistency* is defined by restricting attention to two-agent subgroups.

Bilateral consistency: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $N' \subset N$ with $|N'| = 2$, if $x \equiv \varphi(c, E)$, then $x_{N'} = \varphi\left(c_{N'}, \sum_{i \in N'} x_i\right)$.

Suppose that an awards vector x for a problem is such that its restriction to each two-agent group would be chosen by the rule for the problem of dividing between them the sum of their components of x. The second variable-population property says that x should be chosen by the rule for the original problem.

Converse consistency: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $x \in X(c, E)$, if for each $N' \subset N$ with $|N'| = 2$, $x_{N'} = \varphi\left(c_{N'}, \sum_{i \in N'} x_i\right)$, then $x = \varphi(c, E)$.

The lemma is known as the Elevator Lemma.

Elevator Lemma (Thomson [11]) If a rule φ is *bilaterally consistent* and coincides with a *conversely consistent* rule φ' in the two-agent case, then φ coincides with φ' in general.

Proof. See Thomson [11].

Notice that *sustainability* implies *exemption*. However, it is easy to check that the converse is not true. As we show next, provided *consistency* holds, *exemption* implies *sustainability*. Let n be the cardinality of N.

Proposition 1 If a rule satisfies *exemption* and *consistency*, then it satisfies *sustainability*.

Proof. Let φ be a rule satisfying *exemption* and *consistency*. Let $N \in \mathcal{N}$ and $(c, E) \in \mathcal{C}^N$. Let $N_{sus}(c, E) \equiv \left\{ i \in N \,\middle|\, \sum_{j \in N} \min\{c_j, c_i\} \leq E \right\}$. We assume that $N_{sus}(c, E) \neq \emptyset$ since otherwise there is nothing to check. Let $j \equiv \max\{i \in N_{sus}(c, E) \mid \text{for each } k \in N_{sus}(c, E), c_k \leq c_i\}$. Without loss of generality, we assume that $c_1 \leq c_2 \leq \cdots \leq c_n$. It follows that for each $k \in \{1, 2, \ldots, j\}$, $k \in N_{sus}(c, E)$. We show that for each $k \in \{1, 2, \ldots, j\}$, $\varphi_k(c, E) = c_k$. The proof is by induction on k.

Case 1: $\boldsymbol{k = 1}$. Since $j \in N_{sus}(c, E)$, then $\sum_{k=1}^{j-1} c_k + (n - j + 1) c_j \leq E$. Note that $c_1 \leq c_2 \leq \cdots \leq c_n$. Thus, $nc_1 \leq E$. By *exemption*, $\varphi_1(c, E) = c_1$.

Case 2: $\boldsymbol{k > 1}$. By induction hypothesis, suppose that for each $k \in \{1, 2, \ldots, t\}$, $\varphi_k(c, E) = c_k$, where $t \in \mathbb{N}$ such that $t < j$. We show

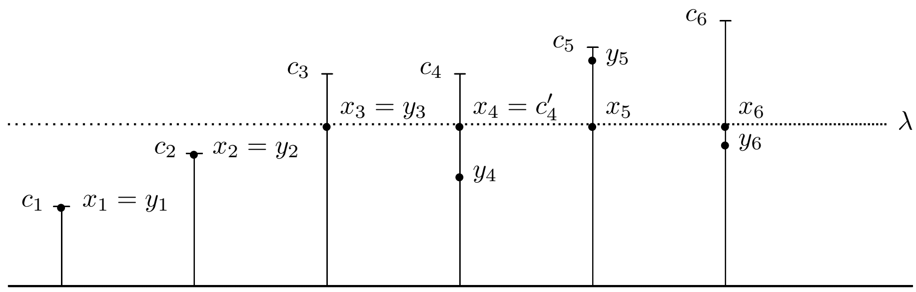

Figure 1: **Illustration of the Proof of Theorem 1.** The claims of agents 1 and 2 are sustainable. Agent 4 receives the smallest amount among the agents whose claims are not sustainable.

that $\varphi_{t+1}(c, E) = c_{t+1}$. Let $N' \equiv \{t+1, t+2, \cdots, n\}$. Since $\sum_{k=1}^{j-1} c_k + (n-j+1)c_j \leq E$ and $c_{t+1} \leq c_{t+2} \leq \cdots \leq c_j \leq \cdots \leq c_n$, then $c_{t+1} \leq \frac{E - \sum_{k=1}^{t} c_k}{n-t}$. By *exemption*, $\varphi_{t+1}\left(c_{N'}, E - \sum_{k=1}^{t} c_k\right) = c_{t+1}$. By induction hypothesis, for each $k \in \{1, 2, \ldots, t\}$, $\varphi_k(c, E) = c_k$. By *consistency*, $\varphi_{t+1}(c, E) = \varphi_{t+1}\left(c_{N'}, E - \sum_{k=1}^{t} c_k\right)$. Thus, $\varphi_{t+1}(c, E) = c_{t+1}$. *Q.E.D.*

3.1 Claims monotonicity

We first investigate the implication of *sustainability* when imposed together with a monotonicity property. This property says that if an agent's claim increases, he should not receive less than what he did initially.[11]

Claims monotonicity: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, each $i \in N$, and each $c'_i \in \mathbb{R}_+$, if $c'_i > c_i$, then $\varphi_i(c'_i, c_{-i}, E) \geq \varphi_i(c, E)$.

All well-known rules satisfy this property. Examples are the "proportional" rule, the constrained equal awards rule, the "constrained equal losses" rule (Dagan [3]), Piniles' rule (Piniles [9]), random arrival rule, and the Talmud rule. Observe that among them, the constrained equal awards rule is the only rule satisfying *sustainability*. In fact, as we show next, *sustainability* and *claims monotonicity* altogether are satisfied only by this rule.

[11]By the notation (c'_i, c_{-i}), we mean the claims vector c in which the i-th component has been replaced by c'_i and $c_{-i} \equiv (c_j)_{j \in N \setminus \{i\}}$.

Theorem 1 The constrained equal awards rule is the only rule satisfying *sustainability* and *claims monotonicity*.

Proof. (Figure 1) Obviously, the constrained equal awards rule satisfies the two properties. Conversely, let φ be a rule satisfying the properties. Let $N \in \mathcal{N}$ and $(c, E) \in \mathcal{C}^N$. Without loss of generality, we assume that $c_1 \leq c_2 \leq \cdots \leq c_n$. Let $N_{sus}(c, E) \equiv \left\{i \in N \,\middle|\, \sum_{j \in N} \min\{c_j, c_i\} \leq E\right\}$, $x \equiv CEA(c, E)$, and $y \equiv \varphi(c, E)$. We show that $x = y$.

Suppose, by contradiction, that $x \neq y$. By *sustainability*, for each $i \in N_{sus}(c, E)$, $y_i = c_i$. Thus, for each $i \in N_{sus}(c, E)$, $y_i = x_i$. Note that for each $i \in N \backslash N_{sus}(c, E)$, $x_i = \lambda < c_i$ where λ is such that $\sum_{i \in N} x_i = E$. Since $\sum_{i \in N} x_i = \sum_{i \in N} y_i$ and $x \neq y$, there is $k \in N \backslash N_{sus}(c, E)$ such that $y_k < \lambda < c_k$. Let $j \in N \backslash N_{sus}(c, E)$ be such that $y_j = y^{\min} \equiv \min_{i \in N \backslash N_{sus}(c,E)} y_i$. Since $y_j \leq y_k$ and $j \in N \backslash N_{sus}(c, E)$, then $y_j < \lambda < c_j$. Let c' be such that $c'_j \equiv \lambda$ and $c'_{-j} \equiv c_{-j}$. Note that $y_j < c'_j$ and $y_{-j} \leqq c'_{-j}$. Thus, $E = \sum_{i \in N} y_i < \sum_{i \in N} c'_i$. It follows that (c', E) is well-defined. Note that $\sum_{i \in N} \min\left\{c'_j, c'_i\right\} = E$. By *sustainability*, $\varphi_j(c', E) = c'_j = \lambda$. By *claims monotonicity*, $\varphi_j(c', E) \leq \varphi_j(c, E)$. Thus, $\lambda \leq y_j$ in violation of $\lambda > y_j$. *Q.E.D.*

Notice that *exemption* is equivalent to *sustainability* in the two-agent case, and that the constrained equal awards rule is *bilaterally consistent* and *conversely consistent*. With these facts, the next two results are immediate consequences of Theorem 1 and the Elevator Lemma together.

Proposition 2 The constrained equal awards rule is the only rule satisfying *exemption*, *claims monotonicity*, and *bilateral consistency*.

Proof. Obviously, the constrained equal awards rule satisfies the three properties. Conversely, let φ be a rule satisfying the properties. Note that *exemption* is equivalent to *sustainability* in the two-agent case. Thus, Theorem 1 implies that $\varphi = CEA$ in that case. Note that φ is *bilaterally consistent*, and that the constrained equal awards rule is *conversely consistent*. By the Elevator Lemma, $\varphi = CEA$ in general. *Q.E.D.*

Proposition 3 The constrained equal awards rule is the only rule satisfying *exemption*, *claims monotonicity*, and *converse consistency*.

Proof. Obviously, the constrained equal awards rule satisfies the three properties. Conversely, let φ be a rule satisfying the properties. Note that *exemption* is equivalent to *sustainability* in the two-agent case. Thus, Theorem 1 implies that $\varphi = CEA$ in that case. Note that φ is *conversely consistent*, and that the constrained equal awards rule is *bilaterally consistent*. By the Elevator Lemma, $\varphi = CEA$ in general. *Q.E.D.*

3.2 Super-modularity

We next switch our attention to an order property. It says that when the amount available increases, of two agents, the one with the larger claim should not receive a smaller share of the increment than the other (Dagan, Serrano, and Volij [4]).

Super-modularity: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, each $E < E'$, and each pair $\{i, j\} \subseteq N$, if $\sum_{i \in N} c_i \geq E'$ and $c_i \leq c_j$, then $\varphi_i(c, E') - \varphi_i(c, E) \leq \varphi_j(c, E') - \varphi_j(c, E)$.

All rules we have mentioned in Section 3.1 satisfy this property. However, there is no logical relation between *claims monotonicity* and *super-modularity*. The "constrained egalitarian rule" (Chun, Schummer, and Thomson [2]) satisfies the former but not the latter. The following rule satisfies the latter but not the former. Given $N \in \mathcal{N}$ and $(c, E) \in \mathcal{C}^N$ with $c_1 \leq c_2 \leq \cdots \leq c_n$, if $N \equiv \{i, j\}$ and $c_j = 2c_i$, then we apply the Talmud rule; otherwise, we apply the proportional rule.

Note that *super-modularity* implies another order property defined next. It says that of two agents, the one with the larger claim should not receive less than the other. Also, his loss should not be less than the other's (Aumann and Maschler [1]).

Order preservation: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each pair

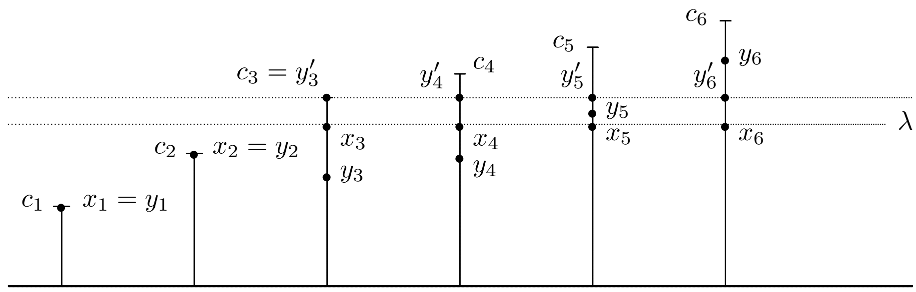

Figure 2: **Illustration of the Proof of Theorem 2.** The claims of agents 1 and 2 are sustainable. Agent 2 receives the largest amount among the agents whose claims are sustainable.

$\{i, j\} \subseteq N$, if $c_i \leq c_j$, then $\varphi_i(c, E) \leq \varphi_j(c, E)$. Also, $c_i - \varphi_i(c, E) \leq c_j - \varphi_j(c, E)$.

We use this implication to prove the next result.

Theorem 2 The constrained equal awards rule is the only rule satisfying *sustainability* and *super-modularity*.

Proof. (Figure 2) Obviously, the constrained equal awards rule satisfies these properties. Conversely, let φ be a rule satisfying the properties. Let $N \in \mathcal{N}$ and $(c, E) \in \mathcal{C}^N$. Without loss of generality, we assume that $c_1 \leq c_2 \leq \cdots \leq c_n$. Let $N_{sus}(c, E) \equiv \left\{ i \in N \,\middle|\, \sum_{j \in N} \min\{c_j, c_i\} \leq E \right\}$, $x \equiv CEA(c, E)$, and $y \equiv \varphi(c, E)$. We show that $x = y$.

Suppose, by contradiction, that $x \neq y$. By *sustainability*, for each $i \in N_{sus}(c, E)$, $y_i = c_i$. Thus, for each $i \in N_{sus}(c, E)$, $x_i = y_i$. Let $k \equiv \max\{i \in N_{sus}(c, E) \mid \text{for each } j \in N_{sus}(c, E),\, y_j \leq y_i\}$. Note that $c_k < c_{k+1} \leq \cdots \leq c_n$. Since *super-modularity* implies the first part of *order preservation*, $y_k \leq y_{k+1} \leq y_{k+2} \leq \cdots \leq y_n$. Note that for each $i \in N_{sus}(c, E)$, $y_i = x_i$, and that for each $i \in N \backslash N_{sus}(c, E)$, $x_i = \lambda < c_i$ where λ is such that $\sum_{i \in N} x_i = E$. Since $\sum_{i \in N} x_i = \sum_{i \in N} y_i$ and $x \neq y$, it follows that $y_{k+1} < \lambda < c_{k+1}$, and that there is $j > k+1$ such that $\lambda < y_j \leq c_j$. Thus, $y_{k+1} < y_j$. Let $E' \equiv \sum_{i \in N_{sus}(c,E)} c_i + |N \backslash N_{sus}(c, E)|\, c_{k+1}$. Since $c_{k+1} \leq c_{k+2} \leq \cdots \leq c_n$, then $E' \leq \sum_{i \in N} c_i$. Thus, (c, E') is well-defined. Since for each $i \in N \backslash N_{sus}(c, E)$, $\lambda < c_i$, then $E < E'$. Let $y' \equiv \varphi(c, E')$. Note that $\sum_{i \in N} \min\{c_i, c_{k+1}\} = E'$, and that $c_1 \leq c_2 \leq \cdots \leq c_n$. By *sustainability*, for each $i \leq k+1$, $y'_i = c_i$. Note that $c_{k+1} \leq c_j$. By the first part of *order preservation*, $y'_{k+1} \leq y'_j$. Recall that $\sum_{i \in N} y'_i = E'$, and that for each $i \leq k+1$, $y'_i = c_i$. It follows that $y'_j = y'_{k+1} = c_{k+1}$. Thus, when the amount available increases from E to E', the increments of agents $k+1$ and j are $y'_{k+1} - y_{k+1} = c_{k+1} - y_{k+1}$ and $y'_j - y_j = c_{k+1} - y_j$. Recall that $y_{k+1} < y_j$. It follows that $y'_{k+1} - y_{k+1} > y'_j - y_j$ in violation of *super-modularity*. *Q.E.D.*

Again, the next two propositions are immediate consequences of Theorem 2 and the Elevator Lemma together. We state them without proofs.

Proposition 4 The constrained equal awards rule is the only rule satisfying *exemption*, *super-modularity*, and *bilateral consistency*.

Proposition 5 The constrained equal awards rule is the only rule satisfying *exemption*, *super-modularity*, and *converse consistency*.

3.3 Order preservation

In the proof of Theorem 2, we use the fact that *super-modularity* implies *order preservation*. One may wonder whether *super-modularity* can be replaced with *order preservation* in Theorem 2, Propositions 4 and 5. The answer is no except in Proposition 4. The following example demonstrates that if we replace *super-modularity* with *order preservation* in Theorem 2, the constrained equal awards rule is not the only acceptable rule.

Example 1 Given $N \in \mathcal{N}$ and $(c, E) \in \mathcal{C}^N$ with $c_1 \leq c_2 \leq \cdots \leq c_n$,

$$\varphi^*(c, E) \equiv \begin{cases} (0, E) & \text{if } N \equiv \{i, j\},\ c_i < c_j, \text{ and } E \leq c_i, \\ (E - c_i, c_i) & \text{if } N \equiv \{i, j\},\ c_i < c_j, \text{ and } c_i < E \leq 2c_i, \\ (c_i, E - c_i) & \text{if } N \equiv \{i, j\},\ c_i < c_j, \text{ and } 2c_i < E \leq c_i + c_j, \\ CEA(c, E) & \text{otherwise.} \end{cases}$$

The next example is a rule that differs from the constrained equal awards rule and satisfies *exemption*, *order preservation*, and *converse consistency*.

Example 2 Given $N \in \mathcal{N}$ and $(c, E) \in \mathcal{C}^N$ with $c_1 \leq c_2 \leq \cdots \leq c_n$,

$$\varphi^{**}(c, E) \equiv \begin{cases} \varphi^*(c, E) & \text{if } N \equiv \{1, 2\}, \\ CEA(c, E) & \text{otherwise}. \end{cases}$$

However, as we show next, if we replace *super-modularity* with *order preservation* in Proposition 4, the constrained equal awards rule is still the only rule satisfying *exemption*, *order preservation*, and *bilateral consistency*. Thus, Proposition 4 can be seen as a corollary of the next result.

Theorem 3 The constrained equal awards rule is the only rule satisfying *exemption*, *order preservation*, and *bilateral consistency*.

Proof. Obviously, the constrained equal awards rule satisfies the three properties. The proof of uniqueness is in two steps. Step 1 establishes that in the two-agent case, if a rule satisfies the three properties, then it is the constrained equal awards rule. Step 2 completes the proof by applying Step 1 and the Elevator Lemma. Let φ be a rule satisfying the three properties.

Step 1: For each $\boldsymbol{N \in \mathcal{N}}$ with $\boldsymbol{|N| = 2}$ and each $\boldsymbol{(c, E) \in \mathcal{C}^N}$, $\boldsymbol{\varphi(c, E) = CEA(c, E)}$.

Let $N \equiv \{i, j\} \in \mathcal{N}$ and $(c, E) \in \mathcal{C}^N$. Without loss of generality, we assume that $c_i \leq c_j$. Let $N_{sus}(c, E) \equiv \left\{ i \in N \;\middle|\; \sum_{j \in N} \min\{c_j, c_i\} \leq E \right\}$, $x \equiv CEA(c, E)$, and $y \equiv \varphi(c, E)$. We show that $x = y$. If $|N_{sus}(c, E)| = 2$, then $E = \sum_{k \in N} c_k$. It follows that $x = y$. If $|N_{sus}(c, E)| = 1$, then $2c_i \leq E < c_i + c_j$. By *exemption*, $y_i = c_i$. Thus, $y_i = x_i$. Since $\sum_{k \in N} x_k = \sum_{k \in N} y_k = E$, then $y_j = x_j$. If $|N_{sus}(c, E)| = 0$, then $x_i = x_j = \frac{E}{2}$. Let $l \in \mathbb{N} \backslash N$ be an agent with claim $c_l \equiv \frac{E}{2}$. Let $N' \equiv \{l, i, j\}$, $c' \equiv (c_l, c_i, c_j)$ and $E' \equiv E + \frac{E}{2}$. Since $\sum_{k \in N' \backslash \{l\}} c'_k = c_i + c_j \geq E$ and $c'_l = c_l = \frac{E}{2}$, then (c', E') is well-defined. Note that $c'_l = \frac{E}{2} = \frac{E'}{3}$. By *exemption*, $\varphi_l(c', E') = \frac{E'}{3}$. Since $|N_{sus}(c, E)| = 0$, then $\frac{E}{2} < c_i = c'_i$. Thus, $c'_l < c'_i \leq c'_j$. By *order preservation*, $\varphi_l(c', E') \leq \varphi_i(c', E') \leq \varphi_j(c', E')$. Note that $\sum_{k \in N'} \varphi_k(c', E') = E'$, and that $\varphi_l(c', E') = \frac{E'}{3}$. Thus, $\varphi_i(c', E') =$

$\varphi_j(c', E') = \frac{E'}{3}$. When agent l leaves with his award $\frac{E'}{3}$, the reduced problem with respect to $\{i, j\}$ and $\varphi(c', E')$ is equivalent to (c, E). By *bilateral consistency*, $\varphi_i(c', E') = \varphi_i\left(c'_{\{i,j\}}, E' - \frac{E'}{3}\right) = \varphi_i(c, E)$ and $\varphi_j(c', E') = \varphi_i\left(c'_{\{i,j\}}, E' - \frac{E'}{3}\right) = \varphi_j(c, E)$. Since $\varphi_i(c', E') = \varphi_j(c', E') = \frac{E'}{3} = \frac{E}{2}$, then $x = y$.

Step 2: Completion of the proof.

Notice that the constrained equal awards rule is *conversely consistent*. By Step 1 and the Elevator Lemma, we conclude that $\varphi = CEA$. *Q.E.D.*

3.4 Independence of the properties

Here, we discuss what additional rules would be made possible by removing one property at a time from the list appearing in each of previous results. For this purpose, we introduce the following rules. The first rule assigns amounts so as to equate the losses experienced by all agents subject to no one receiving a negative amount.

Constrained equal losses rule, *CEL*: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $i \in N$, $CEL_i(c, E) \equiv \max\{0, c_i - \lambda\}$, where λ is chosen such that $\sum_{i \in N} CEL_i(c, E) = E$.

The second one assigns equal amounts to the agents with the smallest claim until they are fully compensated. The remainder is then divided similarly to the agents with the second smallest claim, and so on.

Example 3 (Herrero and Villar [5]) Let $\boldsymbol{\varphi'}$ be defined as follows. Let $N \in \mathcal{N}$, $(c, E) \in \mathcal{C}^N$, and $N_0 \equiv \emptyset$. Given $k \in \mathbb{N}$ such that $1 \leq k \leq |N|$, let $N_k(c) \equiv \left\{ i \in N \left| c_i = \min_{j \in N \setminus \cup_{s<k} N_s(c)} c_j \right. \right\}$ and $y^k \equiv \min_{j \in N \setminus \cup_{s<k} N_s(c)} c_j$. Now, for each $i \in N_k(c)$,

$$\varphi'_i(c, E) \equiv \begin{cases} 0 & \text{if } 0 \leq E \leq \sum_{s<k} |N_s(c)| \, y^s, \\ \frac{E - \sum_{s<k} |N_s(c)| y^s}{|N_k(c)|} & \text{if } \sum_{s<k} |N_s(c)| \, y^s < E \leq \sum_{s \leq k} |N_s(c)| \, y^s, \\ c_i & \text{otherwise.} \end{cases}$$

Property \ Rule	CEL	φ'	φ''	φ^{**}
sustainability	No	Yes	No	Yes
exemption	No	Yes	Yes	Yes
claim monotonicity	Yes	No	Yes	No
super-modularity	Yes	No	Yes	No
order preservation	Yes	No	Yes	Yes
bilateral consistency	Yes	Yes	No	No
converse consistency	Yes	Yes	No	Yes

Table 1: Independence of the properties in each of our results

The last one consists of two parts. The first part deals with the situations in which the number of agents is 3 and agents' claims are different. When equal division is at most as large as the smallest claim, the rule assigns equal amounts to all agents. When equal division is greater than the smallest claim, the rule fully reimburses the agent with the smallest claim. Then the agent with the second smallest claim receives equal division plus one-third of the difference between the equal division and the smallest claim. The last agent then receives the remainder. Continue this procedure subject to no one receiving more than his claim; otherwise, the constrained equal awards rule is applied.

Example 4 (Yeh [14]) Let $\boldsymbol{\varphi''}$ be defined as follows. Let $N \in \mathcal{N}$ and $(c, E) \in \mathcal{C}^N$. Without loss of generality, we assume that $c_1 \leq c_2 \leq \cdots \leq c_n$. Then,

$$\varphi''(c, E) \equiv \begin{cases} \varphi^{***}(c, E) & \text{if } |N| = 3 \text{ and for each pair } \{i, j\} \subset N,\ c_i \neq c_j, \\ CEA(c, E) & \text{otherwise.} \end{cases}$$

where φ^{***} is defined as follows: let $N \equiv \{i, j, k\}$ and $(c, E) \in \mathcal{C}^N$ with $c_i < c_j < c_k$.

$$\varphi^{***}(c, E) \equiv \begin{cases} \left(\frac{E}{3}, \frac{E}{3}, \frac{E}{3}\right) & \text{if } \frac{E}{3} \leq c_i, \\ \left(c_i, \frac{E}{3} + \frac{1}{3}\left(\frac{E}{3} - c_i\right), \frac{E}{3} + \frac{2}{3}\left(\frac{E}{3} - c_i\right)\right) & \text{if } c_i < \frac{E}{3} \leq \frac{c_i + 3c_j}{4}, \\ \left(c_i, c_j, E - c_i - c_j\right) & \text{otherwise.} \end{cases}$$

Table 1 shows that the properties appearing in each of our results are independent. For example, φ' satisfies *sustainability* but not *claims monotonicity*. The constrained equal losses rule satisfies *claims monotonicity* but not *sustainability*. Thus, the properties in Theorem 1 are independent.

4 Dual results

In the literature on axiomatic claims problems, the dual of a characterization can be derived by exploiting dual relations between rules, and between properties of rules. Given a rule φ, its dual, denoted by φ^d, is obtained by first replacing the amount available with its "complement" (the difference between the sum of the claims and itself), then applying φ to distribute that difference, and finally subtracting the resulting awards vector from the claims vector. Formally, for each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$, $\varphi^d(c, E) \equiv c - \varphi(c, \sum_{i \in N} c_i - E)$. Clearly, the constrained equal awards rule and the constrained equal losses rule are dual of each other (Herrero and Villar [6]).

Similarly, any property can also be associated with its dual. We say that two properties are dual if whenever a rule satisfies one of them, its dual satisfies the other.[12] The dual of *sustainability* and *exemption* are "independence of residual claims" and "exclusion", respectively (Herrero and Villar [5,6]). The dual of *claims monotonicity* is formulated by Thomson and Yeh [13]. Examples of the properties that are dual of themselves are *super-modularity* (Thomson [12]), *bilateral consistency* (Herrero and Villar [6]), *converse consistency* and *order preservation* (Thomson and Yeh [13]).

Thus, the dual of Theorems 1, 2, and 3, are that the constrained equal losses rule is the only rule satisfying *independence of residual claims* and *the dual of claims monotonicity* (the dual of Theorem 1) or *super-modularity* (the dual of Theorem 2), and that it is the only rule satisfying

[12] For a study of the dual relations between rules, and the properties of rules, see Thomson and Yeh [13].

exclusion, *order preservation*, and *bilateral consistency* (the dual of Theorem 3). Similarly, the dual of Propositions 2, 3, 4, and 5 are that the constrained equal losses rule is the only rule satisfying *exclusion*, *the dual of claims monotonicity*, and *bilateral consistency* (the dual of Proposition 2) or *converse consistency* (the dual of Proposition 3), and that it is the only rule satisfying *exclusion*, *super-modularity*, and *bilateral consistency* (the dual of Proposition 4) or *converse consistency* (the dual of Proposition 5).

5 Extensions

We extend the ideas of *sustainability* and *exemption* to groups of agents. We say that the claims of a group of agents are "group sustainable" if truncating all other claims at the arithmetic average of claims of that group results in a situation where there is enough to fully compensate everyone. The property, *group sustainability*, says that if the claims of a group of agents are group sustainable, each agent in that group should be fully compensated.

Group sustainability: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $N' \subset N$, if $\sum_{j \in N'} c_j + \sum_{j \in N \setminus N'} \min \left\{ c_j, \frac{\sum_{i \in N'} c_i}{|N'|} \right\} \leq E$, then $\varphi_{N'}(c, E) = c_{N'}$.

We show that such an extension of *sustainability* is extremely demanding.

Theorem 4 No rule satisfies *group sustainability*.

Proof. The proof is by means of an example. Let φ be a rule satisfying the property. Let $N \equiv \{1, 2, 3\}$, $c \equiv (1, 5, 5)$, and $E \equiv 9$. Let $x \equiv \varphi(c, E)$. Let $N' \equiv \{1, 2\}$ and $N'' \equiv \{1, 3\}$. Note that $\frac{\sum_{i \in N'} c_i}{|N'|} = 3$ and $\sum_{j \in N'} c_j + \sum_{j \in N \setminus N'} \min \left\{ c_j, \frac{\sum_{i \in N'} c_i}{|N'|} \right\} = 9 = E$. By *group sustainability*, $x_1 = 1$ and $x_2 = 5$. Similarly, $x_1 = 1$ and $x_3 = 5$. Thus, $\sum_{i \in N} x_i = 11 > 9$ in violation of $\sum_{i \in N} x_i = 9$. *Q.E.D.*

The intuition of Theorem 4 is that *group sustainability* may protect agents whose claims are not sustainable. For instance, in the proof of this theorem, the claim of agent 2 is not sustainable. However, when he forms a group with agent 1, their claims are group sustainable. Then, agent 2 is fully reimbursed by *group sustainability*, but not by *sustainability*. The same reasoning applies for agent 3. Thus, all agents will be fully compensated. That is impossible.

Next, we formulate a version of *exemption* for groups. We say that the claims of a group of agents are "group exemptive" if the arithmetic average of claims of this group is not greater than equal division. The property, *group exemption*, says that if the claims of a group of agents are group exemptive, each agent in that group should be fully reimbursed.

Group exemption: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $N' \subset N$, if $\frac{\sum_{i \in N'} c_i}{|N'|} \leq \frac{E}{|N|}$, then $\varphi_{N'}(c, E) = c_{N'}$.

As we show next, such an extension of *exemption* is very demanding too. The intuition of this result is similar to that of Theorem 4.

Theorem 5 No rule satisfies *group exemption*.

Proof. The proof is by means of an example. Let φ be a rule satisfying the property. Let $N \equiv \{1, 2, 3\}$, $c \equiv (1, 4, 5)$, and $E \equiv 9$. Let $N' \equiv \{1, 2\}$ and $N'' \equiv \{1, 3\}$. Let $x \equiv \varphi(c, E)$. Note that $\frac{\sum_{i \in N'} c_i}{|N'|} = \frac{5}{2}$ and $\frac{E}{3} = 3$. By *group exemption*, $x_1 = 1$ and $x_2 = 4$. Similarly, $x_1 = 1$ and $x_3 = 5$. Thus, $\sum_{i \in N} x_i = 10 > 9$ in violation of $\sum_{i \in N} x_i = 9$. *Q.E.D.*

6 Conclusion

We provided a systematic analysis of *sustainability* and *exemption* when imposed together with other natural properties, and showed that *sustainability* and *exemption* have strong implications. That is, in the presence of very mild properties such as *claims monotonicity*, *super-modularity*, or *order preservation*, only one rule satisfies *sustainability* and *exemption* separately, and this rule is the constrained equal awards rule. These results furthered our understanding of the rule, and confirmed the importance it has played in recent work. In addition, we extended the ideas of *sustainability* and *exemption* to groups of agents, and found that no rule satisfies each of these extensions. These impossibility results suggested that taking the arithmetic average of claims of a group of agents to extend the ideas of the protective properties for groups is too demanding.

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Appendix that is not part of the submission for publication

To save space, we have included in this appendix, which is not for publication, formal definitions of certain rules and certain properties that play auxiliary roles. We begin with formal definitions of the proportional rule, the Talmud rule, the Piniles' rule, the random arrival rule, and the constrained egalitarian rule.

The proportional rule assigns awards proportional to claims.

Proportional rule, $\boldsymbol{P}$: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $i \in N$, $P_i(c, E) \equiv \lambda c_i$, where λ is chosen such that $\sum_{i \in N} P_i(c, E) = E$.

The Talmud rule is defined by Aumann and Maschler [1] to rationalize the recommendations made in the Talmud for several numerical examples. It is a hybrid of the constrained equal awards and constrained equal losses rules.

Talmud rule, $\boldsymbol{T}$: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $i \in N$,

$$T_i(c, E) \equiv \begin{cases} \min\left\{\frac{c_i}{2}, \lambda\right\} & \text{if } \sum \frac{c_i}{2} \geq E, \\ \max\left\{\frac{c_i}{2}, \frac{c_i}{2} - \lambda\right\} & \text{otherwise.} \end{cases}$$

where λ is chosen such that $\sum_{i \in N} T_i\,(c, E) = E$.

The Piniles' rule (Piniles [9]) can be understood as resulting from the "twice" application of the constrained equal awards rule.

Piniles' rule, $\boldsymbol{Pin}$: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $i \in N$,

$$Pin_i(c, E) \equiv \begin{cases} \min\left\{\frac{c_i}{2}, \lambda\right\} & \text{if } \sum \frac{c_i}{2} \geq E, \\ \frac{c_i}{2} + \min\left\{\frac{c_i}{2}, \lambda\right\} & \text{otherwise.} \end{cases}$$

where λ is chosen such that $\sum_{i \in N} Pin_i\,(c, E) = E$.

The random arrival rule (O'Neill [7]) is defined on the basis of first-come first-serve scheme associated with any particular order in which agents arrive, let us take the arithmetic average of the awards vectors calculated in this way when all orders of arrival are equally probable. Given $N \in \mathcal{N}$, let Π^N designate the class of bijections on N.

Random arrival rule, ***RA***: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $i \in N$,

$$RA_i(c, E) \equiv \frac{1}{|N|!} \sum_{\pi \in \Pi^N} \min \left\{ c_i, \max \left\{ E - \sum_{j \in N, \pi(j) < \pi(i)} c_j, 0 \right\} \right\}.$$

The constrained egalitarian rule (Chun, Schummer, and Thomson [2]) is defined as follows: assume that $c_1 \leq c_2 \leq \cdots \leq c_n$. For amounts available up to $\sum \frac{c_i}{2}$, awards are computed as for the Talmud rule. At that point, any additional unit goes to agent 1 until he receives his claim or half of the second smallest claim, whichever is smaller. If $c_1 \leq \frac{c_2}{2}$, he stops at c_1. If $c_1 > \frac{c_2}{2}$, any additional unit is divided equally between agents 1 and 2 until they reach c_1, at which point agent 1 drops out, or they reach $\frac{c_3}{2}$. In the first case, any additional unit goes entirely to agent 2 until he reaches c_2 or $\frac{c_3}{2}$. In the second case, any additional unit is divided equally between agents 1, 2, and 3 until they reach c_1, at which point agent 1 drops out, or they reach $\frac{c_4}{2}$, and so on.

Constrained egalitarian rule, ***CE***: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $i \in N$,

$$CE_i(c, E) \equiv \begin{cases} \min\left\{\frac{c_i}{2}, \lambda\right\} & \text{if } E \leq \sum \frac{c_j}{2}, \\ \max\left\{\frac{c_i}{2}, \min\left\{c_i, \lambda\right\}\right\} & \text{otherwise.} \end{cases}$$

where λ is chosen such that $\sum_{i \in N} CE_i(c, E) = E$.

Next is the formal definition of *composition down* (Moulin [8]).

Composition down: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $0 \leq E' \leq E$, we have $\varphi(c, E') = \varphi(\varphi(c, E), E')$.

Now, we formally define the dual properties of *sustainability*, *exemption*, and *claims monotonicity*. We begin with the dual of *sustainability*. We say that agent i's claim is "residual" if the aggregate excess claim relative to this agent exceeds the worth of the amount available, namely $E \leq \sum_{j \in N} \max\{0, c_j - c_i\}$. *Independence of residual claims* requires that if an agent's claim is residual, he should get nothing.

Independence of residual claims: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $i \in N$, if $E \leq \sum_{j \in N} \max\{0, c_j - c_i\}$, then $\varphi_i(c, E) = 0$.

Next is the dual of *exemption*. We say that agent i's claim is "exclusive" if his claim is not greater than the average loss, namely, $c_i \leq \frac{\sum_{j \in N} c_j - E}{|N|}$. *Exclusion* says that if agent i's claim is exclusive, he should get nothing.

Exclusion: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $i \in N$, if $c_i \leq \frac{\sum_{j \in N} c_j - E}{|N|}$, then $\varphi_i(c, E) = 0$.

The dual of *claims monotonicity* follows. It says that if an agent's claim and the amount available increase by the same amount α, this agent's award should not increase by more than α.

Dual of claims monotonicity: For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, each $i \in N$, and $\alpha \in \mathbb{R}_+$, we have $\varphi_i(c_i + \alpha, c_{-i}, E + \alpha) - \varphi_i(c, E) \leq \alpha$.

In the context, we claim that φ^{**} is *conversely consistent*. Here we provide a proof. The proof makes use of the facts that φ^{**} satisfies "resource monotonicity"[13] and *order preservation*. These facts are immediate consequences of the definition of the rule.

Claim 1 φ^{**} is *conversely consistent*.

[13]This property says that if the amount available increases, no one should receive less than what he did initially. Formally, for each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $E' > E$, if $\sum_{i \in N} c_i \geq E'$, then $\varphi(c, E') \geq \varphi(c, E)$.

Proof. Let $N \in \mathcal{N}$ and $(c, E) \in \mathcal{C}^N$. Without loss of generality, we assume that $c_1 \le c_2 \le \cdots \le c_n$. Let $x \in X(c, E)$ be such that for each $N' \subset N$ with $|N'| = 2$, $x_{N'} = \varphi^{**}\left(c_{N'}, \sum_{i \in N'} x_i\right)$. Let $y \equiv \varphi^{**}(c, E)$. We assume that $|N| \ge 3$ since otherwise there is nothing to check. By the definition of φ^{**}, $y \equiv CEA(c, E)$. We show that $x = y$.

Step 1: For each $k \in N \setminus \{1, 2\}$, $x_k = y_k$. Suppose, by contradiction, that there exists $k \in N \setminus \{1, 2\}$ such that $x_k \ne y_k$. Without loss of generality, let $x_k > y_k$. Since $\sum_{i \in N} x_i = \sum_{i \in N} y_i$, then there exists $k' \in N \setminus \{k\}$ such that $x_{k'} < y_{k'}$. Note that $\{k, k'\} \ne \{1, 2\}$. By the definition of φ^{**}, we have to apply the constrained equal awards rule to the situations involving only agents k and k'. If $x_k + x_{k'} \ge y_k + y_{k'}$, then since CEA satisfies *resource monotonicity*, it follows that $(x_k, x_{k'}) = CEA(c_k, c_{k'}; x_k + x_{k'}) \ge CEA(c_k, c_{k'}; y_k + y_{k'}) = (y_k, y_{k'})$. Thus, $x_{k'} \ge y_{k'}$ in violation of $x_{k'} < y_{k'}$. If $x_k + x_{k'} < y_k + y_{k'}$, then since CEA satisfies *resource monotonicity*, it follows that $(x_k, x_{k'}) = CEA(c_k, c_{k'}; x_k + x_{k'}) \le CEA(c_k, c_{k'}; y_k + y_{k'}) = (y_k, y_{k'})$. Thus, $x_k \le y_k$ in violation of $x_k > y_k$.

Step 2: $x_1 = y_1$ and $x_2 = y_2$. Suppose, by contradiction, that $x_1 \ne y_1$. We consider two cases.

Case 1: $x_1 > y_1$. By Step 1 and the fact that $\sum_{i \in N} x_i = \sum_{i \in N} y_i$, $x_1 + x_2 = y_1 + y_2$. Thus, $x_2 < y_2$. We consider three subcases.

Subcase 1.1: $x_1 + x_2 \le c_1$. By the definition of φ^*, $x_1 = 0$. Since $x_1 > y_1$, it follows that $y_1 < 0$ in violation of $y_1 \ge 0$.

Subcase 1.2: $c_1 < x_1 + x_2 \le 2c_1$. By the definition of φ^*, $x_2 = c_1$. Let $k \in N \setminus \{1, 2\}$. Note that $(x_2, x_k) = CEA(c_2, c_k; c_1 + x_k)$. Since $c_1 \le c_2 \le c_k$ and $x_2 = c_1$, then $x_k = \frac{c_1 + x_k}{2}$. Thus, $x_k = c_1$. By Step 1, $y_k = x_k$. Thus, $y_k = c_1$. Note that $y_2 > x_2 = c_1$ and $y_k = c_1$. It follows that $y_k < y_2$. Since $c_2 \le c_k$ and φ^{**} satisfies *order preservation*, then $y_k \ge y_2$ in violation of $y_k < y_2$.

Subcase 1.3: $2c_1 < x_1 + x_2 \le c_1 + c_2$. Let $k \in N \setminus \{1, 2\}$. Note that $(x_2, x_k) = CEA(c_2, c_k; x_2 + x_k)$. Since $c_2 \le c_k$, then either $x_2 = c_2$ or

$x_2 = x_k$ $\left(x_2 = \frac{x_2+x_k}{2}\right)$. If $x_2 = c_2$, then since $x_2 < y_2$, it follows that $c_2 < y_2$ in violation of $y_2 \leq c_2$. If $x_2 = x_k$, then by Step 1, $y_k = x_k$. Thus, $y_k = x_2$. Since $x_2 < y_2$, it follows that $y_k < y_2$. Since $c_2 \leq c_k$ and φ^{**} satisfies *order preservation*, then $y_2 \leq y_k$ in violation of $y_k < y_2$.

Case 2: $\boldsymbol{x_1 < y_1}$. We consider two subcases.

Subcase 2.1: $x_1 + x_2 \leq 2c_1$. By the definition of φ^*, $x_1 \leq c_1$. Let $k \in N \setminus \{1, 2\}$. Note that $(x_1, x_k) = CEA\,(c_1, c_k; x_1 + x_k)$. Since $c_1 \leq c_k$ and $x_1 \leq c_1$, then either $x_1 = c_1$ or $x_1 = x_k$ $\left(x_k = \frac{x_1+x_k}{2}\right)$. If $x_1 = c_1$, then since $x_1 < y_1$, it follows that $c_1 < y_1$ in violation of $y_1 \leq c_1$. If $x_1 = x_k$, then by Step 1, $y_k = x_k$. Thus, $y_k = x_1$. Since $x_1 < y_1$, it follows that $y_k < y_1$. Since $c_1 \leq c_k$ and φ^{**} satisfies *order preservation*, then $y_1 \leq y_k$ in violation of $y_1 > y_k$.

Subcase 2.2: $2c_1 < x_1 + x_2 \leq c_1 + c_2$. By the definition of φ^*, $x_1 = c_1$. Since $x_1 < y_1$, it follows that $c_1 < y_1$ in violation of $y_1 \leq c_1$. *Q.E.D.*